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Time Domain Finite Element Analysis of Dynamic Systems

Jinyoung Suk* and Youdan Kim†
Seoul National University, Seoul 151-742, Republic of Korea

In this study, dynamic analyses of systems are conducted using the time domain finite element analysis when initial and final conditions are specified. Dynamics of systems can be analyzed using the time-marching procedure based on Hamilton's weak principle. A new method is developed to describe the motion of a system and is extended to accommodate the initial displacement as a state variable. System matrices are constructed to evaluate time histories of the system from initial time to a specified time without the information on the velocity and momentum at each time step. Galerkin's weak principle is used to construct element matrices in the time domain and assemble the whole matrices. The Neumann boundary condition is modified to reflect the effect of initial displacement and velocity that are inherent to dynamic problems through the momentum conservation of the system. The matrices obtained thereafter conserve the symmetricity and enable us to reduce the model order by the systematic approach. Spatial propagation equation is built up, and two-point boundary conditions are used to estimate the unknown initial conditions at one end of the beam. Modal domain analysis is introduced to reduce the size of matrices. It can be seen that the matrices constructed using the time domain finite element analysis are applicable to the model reduction that is analogous to the space-domain model reduction methodology. Several numerical examples show that a consequence of the suggested method can be applied to describe the motion of various dynamic systems successfully and to validate the effect of the model reduction in the time domain.

Nomenclature

A = system matrix
B = input influence matrix
$d(t)$ = known external disturbance
EI = dynamic stiffness
$\boldsymbol{F}$ = time-based force vector
$f(x,t)$ = force distribution
$\boldsymbol{K}$ = time-based stiffness matrix
L = length of beam
$\boldsymbol{M}$ = time-based mass matrix
M_b = bending moment of a beam
T = kinetic energy
t = time
t_f = a specified final time
$u(t)$ = external force
V = potential energy
V_s = shear force of a beam
$\boldsymbol{v}$ = state vector that is composed of $v_j(x)$
$v(x,t)$ = subsequent motion of a beam released from initial condition
W = virtual work
$\boldsymbol{w}$ = state vector
$w(x,t)$ = displacement of a beam
w_i = generalized spatial distribution
x = spatial coordinate
x_f = position of a force actuator from the root of a beam
$y(x)$ = state vector in state-space formulation
δ_{rs} = Kronecker delta
$\eta(x)$ = modal coordinate
$\theta(x)$ = transverse angle
ρ = density
Φ = modal matrix
ϕ_i = shape function

Subscripts

a = augmented property in method 2
0 = initial displacement

Operators

Ξ = $-\mathrm{d}^2/\mathrm{d}t^2$
Π = -1

Introduction

DYNAMICS of flexible structures are usually governed by the first- and second-order hyperbolic equation and are represented as a function of two individual variables called time and space. A large portion of structural dynamics has been analyzed via the space-discretized finite element method (FEM). Conventional FEM, which fixes the space domain using discretization and treats the time variable as a unique variable, has been applied successfully to the static/dynamic analysis of structures by way of mathematical background and systematic organization of mass and stiffness matrices. Moreover, it has been combined with the linear control theory to deal with the control problems of the structural systems. On the other hand, there exists another structural research field that FEM cannot reach: researches on the spatial distribution problem of various properties.

The method that fixes time and analyzes the system in the space domain is studied by von Flotow and Schafer,[1] Fujii et al.,[2] Fujii and Ohtsuka,[3] Fujii et al.,[4] and MacMartin and Hall.[5] They used Fourier transformation[1] or Laplace transformation[2] to freeze the time domain and derived the spatial propagation equation. In their studies, actuator dynamics is subjected to a boundary condition due to the location of an actuator, and the performance of the wave-absorbing controller is analyzed in the frequency domain.[6,7]

On the other hand, dynamic analysis of the system with initial conditions receded in the field of system analysis, as well as linear system analysis, after the time domain FEM was developed by Bailey[8] based on Hamilton's principle.[9–12] As a result, optimization problems of various dynamic systems have been solved as an alternative way to the classical optimal control theory.[13–17] However, all of the time domain finite element deals with the initial condition problems, which yield the time-marching, step-by-step method, and therefore system responses cannot be evaluated in a single round.

In this study, dynamic analyses of systems are conducted using the time domain finite element analysis when the initial and final conditions are partly specified. A new method is developed to describe the motion of the system, and it is extended to accommodate the initial displacement as a state variable. It is well known that the conventional FEM deals with the Neumann boundary condition in case the

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*Graduate Student, Department of Aerospace Engineering.
†Associate Professor, Department of Aerospace Engineering. E-mail: ydkim@plaza.snu.ac.kr. Senior Member AIAA.

flux at both ends is given.[18] Furthermore, one of the boundary conditions can be regressed to the Dirichlet boundary condition under the assumption that a certain compatibility condition is satisfied. In the case of various dynamics problems, the momentum conservation law corresponds to the compatibility condition. With these relationships, we can analyze the dynamic system using the time domain FEM when the initial displacement/velocity and final velocity are prescribed. The method developed herein is distinguished from the conventional time-marching method and its assembled version in the sense that the constructed matrices conserve the symmetricity and, hence, provide a good environment for model reduction.

A spatial propagation equation is built up, and a two-point boundary value problem is used to derive the initial conditions of four physical properties at one end of the beam. It can be seen from the equation that the physical properties transfer through the beam without violating boundary conditions at both ends. It can be seen from the differential equation that a certain property propagates bilaterally through a medium in the spatial domain, departing from either direction of the beam. It is called the wave because the closed-form matrix differential equation obtained herein preserves the original second-order wave equation of the beam. Modal domain analysis is introduced to reduce the size of the matrix constructed by the time domain FEM. Much research has been done on model reduction in the field of structural control and linear control theory.[19–21] Model reduction in the time domain used herein is the extension of the space domain equivalence. It is verified that the time-based modal coordinate system stays within the region of the self-adjoint system. This enables the reduction of modes, and finally we can reduce the size of the system significantly. This also means that a small number of the modal coordinates in the time domain can well describe the complex system response. Several numerical examples are shown to verify the proposed methods and to validate the effect of significant model reduction.

Time Domain Finite Element Analysis for Dynamic Systems

In this section, two methods using the time domain finite element analysis are developed to describe the motion of dynamic systems. A matrix wave equation is derived in the time domain to depict the motion of a general distributed parameter system. The proposed method to build the system matrices can be directly applied without loss of generality to the dynamics of a particle, a system of particles, or a rigid body coupled with elastic modes.

Method 1: Wave Equation of a Beam

In this section, a spatial propagation equation of a cantilever beam is derived by using the time domain finite element analysis. An extended Hamilton's principle is applied to derive the equation of propagation. Let us consider the transverse vibration of a fixed-free Euler–Bernoulli beam. The kinetic energy of a beam is expressed as follows:

$$T = \frac{1}{2}\int_0^L \rho(x)\dot{w}^2(x,t)\,\mathrm{d}x \tag{1}$$

where $w(x,t)$ denotes the displacement of a beam and L is the length of the beam. Carrying out the variation of integrand in Eq. (1) and integrating the resulting equation over a finite time t_f yields

$$\int_0^{t_f}\delta T\,\mathrm{d}t = \int_0^{t_f}\int_0^L \rho\dot{w}\delta\dot{w}\,\mathrm{d}x\,\mathrm{d}t \tag{2}$$

Similarly, the variation of potential energy can be described after integration as

$$\int_0^{t_f}\delta V\,\mathrm{d}t = \int_0^{t_f}\int_0^L EIw''\delta w''\,\mathrm{d}x\,\mathrm{d}t \tag{3}$$

Integrating Eq. (3) by parts with respect to the spatial domain gives

$$\int_0^{t_f}\delta V\,\mathrm{d}t = \int_0^{t_f}\int_0^L EIw^{(iv)}\delta w\,\mathrm{d}x\,\mathrm{d}t + EIw''\delta w'|_0^L - EIw'''\delta w|_0^L \tag{4}$$

Note that the last two terms in the right-hand side vanish due to the geometric and natural boundary conditions of the beam. The virtual work done by the nonconservative distributed force has the form of

$$\int_0^{t_f}\delta W\,\mathrm{d}t = \int_0^{t_f}\int_0^L f(x)u(t)\delta w(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{5}$$

where $f(x)u(t)$ is an arbitrary distributed external force acting on the beam. The equation of motion can be obtained by substituting Eqs. (2), (4), and (5) into the following extended Hamilton's principle:

$$\int_0^{t_f}\delta(T-V)\,\mathrm{d}t + \int_0^{t_f}\delta W\,\mathrm{d}t = 0 \tag{6}$$

To generate an N-degree-of-freedom differential equation model for a continuous system, we approximate the displacement of the beam and its virtual duplicate at a certain time by

$$w(x,t) = \sum_{i=1}^{N}\phi_i(t)w_i(x), \qquad \delta w(x,t) = \sum_{i=1}^{N}\phi_i(t)\delta w_i(x) \tag{7}$$

where the various $\phi_i(t)$ are shape functions that should be selected to satisfy the prescribed initial and final conditions. Note that the shape functions adopted herein from Eq. (2) are to be selected so that their first time derivatives may exist. The state vector defined by $\boldsymbol{w}(x) = [w_1(x)\ \ w_2(x)\ \ \cdots\ \ w_N(x)]^T$ is the sequence of the beam deflection as a function of a spatial coordinate. After applying the extended Hamilton's principle, we obtained

$$\int_0^L\left[\delta\boldsymbol{w}^T(x)\boldsymbol{M}\boldsymbol{w}(x) - \delta\boldsymbol{w}^T(x)\boldsymbol{K}\boldsymbol{w}^{(iv)}(x) + \delta\boldsymbol{w}^T(x)\boldsymbol{F}f(x)\right]\mathrm{d}x = 0 \tag{8}$$

where

$$\boldsymbol{M}_{ij} = \int_0^{t_f}\rho\dot{\phi}_i(t)\dot{\phi}_j(t)\,\mathrm{d}t \tag{9}$$

$$\boldsymbol{K}_{ij} = \int_0^{t_f}EI\phi_i(t)\phi_j(t)\,\mathrm{d}t \tag{10}$$

$$\boldsymbol{F}_i = \int_0^{t_f}\phi_i(t)u(t)\,\mathrm{d}t \tag{11}$$

The assembled set of differential equations of motion can be expressed as follows:

$$-\boldsymbol{M}\boldsymbol{w}(x) + \boldsymbol{K}\boldsymbol{w}^{(iv)}(x) = \boldsymbol{F}f(x) + \boldsymbol{F}_0 w_0(x) \tag{12}$$

where $w_0(x)$ is the initial displacement of the beam and $\boldsymbol{F}_0$ is generated by the initial deflection of the beam. As can be seen in Eq. (12), the spatial propagation of the beam depends on the spatial distribution of external input. In this formulation, initial and final conditions act like boundary conditions in the space domain. It may be verified that the effect of the initial spatial displacement takes the form of another external input as well as a forcing vector.

In this paper, the Neumann boundary condition is used to deal with the initial and final velocity. However, a rigid mode is entailed in the Neumann boundary condition because velocity is the only condition given at both ends. To remove this, the initial displacement is imposed on the first part of the assembled matrices. This is similar to applying the Dirichlet boundary condition to the initial state and the Neumann boundary condition to the final state. However, it is not always possible to convert the Neumann boundary condition into a mixed form because a certain compatibility condition should be satisfied. It can be regarded as a conservation of equivalent flux throughout the media, and its time domain counterpart is the momentum conservation law in dynamic systems. The compatibility condition holds for the time domain FEM because the dynamic system always preserves the equivalent linear/angular momentum in the sense that the rate of the linear/angular momentum reflects on the sum of the equivalent external force/moment.

Method 2: Wave Equation Including Initial Value in State Vector

In the preceding section, the matrix differential equation of a beam is constructed using the time finite element analysis. Using the developed method, a large class of dynamic systems can be analyzed efficiently, for example, 1) propagation of a beam activated by the actuator input with zero initial displacement, 2) propagation of a beam activated by the actuator input with zero initial displacement for the case where external disturbance is involved, and 3) propagation of a beam activated by the prescribed initial displacement to follow the prescribed external reference to be tracked.

Now, let us propose a method that includes the initial displacement $w_0(x)$ as a variable. To rederive the equations of motion for the whole system, we define a state vector as follows:

$$\boldsymbol{w}_a(x) = [w_0(x) \quad v_1(x) \quad v_2(x) \quad \cdots \quad v_N(x)]^T$$

where $v_j(x)$ denotes the subsequent motion of the beam released from the initial displacement at time t_j. Because the initial displacement is not influenced by the motion taken thereafter, $w_0(x)$ should be kept independent of $v_j(x)$. On the contrary, $v_j(x)$ depends on the initial displacement $w_0(x)$. Therefore, $w_0(x)$ should be coupled with $v_j(x)$ in one direction.

The static deflection of a beam with the distributed load $f_0(x)$ can be described as

$$EIw_0^{(iv)}(x) = f_0(x) \tag{13}$$

With initial displacement $w_0(x)$, the motion of the beam can be governed by the following partial differential equation:

$$\rho\ddot{w}(x,t) + EIw^{(iv)}(x,t) = f(x,t) \tag{14}$$

where $f(x,t)$ is a force distribution with respect to the temporal and spatial coordinates and $w(x,t)$ is the deflection of the beam with respect to the space and time. In this study, we assume that $f(x,t)$ can be represented as $f(x,t) = f(x)u(t)$. Total displacement $w(x,t)$ can be represented by the summations of the initial displacement $w_0(x)$ and the motion of the beam $v(x,t)$ due to the inertial action caused by initial displacement, actuator input, and external disturbances, as follows:

$$w(x,t) = w_0(x) + v(x,t) \tag{15}$$

where $v(x,0) = 0$. All initial conditions except for the initial displacement can be imposed on $\dot{w}(x,0)$ and $\ddot{w}(x,0)$ as follows:

$$\dot{w}(x,t) = \dot{v}(x,t), \qquad \ddot{w}(x,t) = \ddot{v}(x,t) \tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (14) yields

$$\rho\ddot{v}(x,t) + EI\left[w_0^{(iv)}(x) + v^{(iv)}(x,t)\right] = f(x,t) \tag{17}$$

Using Eq. (13) in Eq. (17), we obtain the equation of motion as follows:

$$\rho\ddot{v}(x,t) + EIv^{(iv)}(x,t) = -f_0(x) + f(x,t) \tag{18}$$

To apply Galerkin's weak principle, multiply Eq. (13) by the virtual initial displacement $\delta w_0(x)$ and integrate the resulting equation over a time:

$$\int_0^{t_f} EIw_0^{(iv)}(x)\delta w_0(x)\,\mathrm{d}t = \int_0^{t_f} f_0(x)\delta w_0(x)\,\mathrm{d}t \tag{19}$$

Similarly, multiplying Eq. (18) by the virtual displacement $\delta v(x,t)$ and integrating over a finite time yields

$$\int_0^{t_f}\left[\rho\ddot{v}(x,t)\delta v(x,t) + EIv^{(iv)}(x,t)\delta v(x,t)\right]\mathrm{d}t = \int_0^{t_f}[-f_0(x,t)\delta v(x,t) + f(x,t)\delta v(x,t)]\,\mathrm{d}t \tag{20}$$

Integrating Eq. (20) by parts leads to the following equation:

$$-\int_0^{t_f}\rho\dot{v}(x,t)\delta\dot{v}(x,t)\,\mathrm{d}t + \rho\dot{v}(x,t)\delta v(x,t)\big|_0^{t_f} + \int_0^{t_f} EIv^{(iv)}(x,t)\delta v(x,t)\,\mathrm{d}t = -\int_0^{t_f}\delta v(x,t)\,\mathrm{d}t f_0(x) + \int_0^{t_f}\delta v(x,t)f(x,t)\,\mathrm{d}t \tag{21}$$

As in Eq. (7), the displacement and the virtual displacement can be discretized using the shape functions $\phi(t)$ as

$$v(x,t) = \sum_{i=1}^{N}\phi_i(t)v_i(x), \qquad \delta v(x,t) = \sum_{i=1}^{N}\phi_i(t)\delta v_i(x) \tag{22}$$

Substituting Eq. (22) into Eq. (21) combined with Eq. (19) yields

$$\int_0^L\left\{-\begin{bmatrix}\delta w_0 & \delta\boldsymbol{v}^T\end{bmatrix}\boldsymbol{M}_a\begin{bmatrix}w_0\\ \boldsymbol{v}\end{bmatrix} + \begin{bmatrix}\delta w_0 & \delta\boldsymbol{v}^T\end{bmatrix}\boldsymbol{K}_a\begin{bmatrix}w_0^{(iv)}\\ \boldsymbol{v}^{(iv)}\end{bmatrix}\right\}\mathrm{d}x = \int_0^L\begin{bmatrix}\delta w_0 & \delta\boldsymbol{v}^T\end{bmatrix}\boldsymbol{F}_a\begin{bmatrix}f_0(x)\\ f(x)\end{bmatrix}\mathrm{d}x \tag{23}$$

where

$$\boldsymbol{M}_a = \begin{bmatrix}0 & \boldsymbol{O}^T\\ \boldsymbol{O} & \boldsymbol{M}\end{bmatrix} \tag{24}$$

$$\boldsymbol{K}_a = \begin{bmatrix}\int_0^{t_f} EI\,\mathrm{d}t & \boldsymbol{O}^T\\ \boldsymbol{O} & \boldsymbol{K}\end{bmatrix} \tag{25}$$

$$\boldsymbol{F}_a = \begin{bmatrix}\int_0^{t_f}\mathrm{d}t & O\\ -\int_0^{t_f}\phi_j\,\mathrm{d}t & \boldsymbol{F}\end{bmatrix} \tag{26}$$

The effect of initial displacement is expressed through the forcing term in Eq. (23). Therefore, the following matrix differential equation is obtained by accommodating the initial displacement:

$$-\boldsymbol{M}_a\boldsymbol{w}_a(x) + \boldsymbol{K}_a\boldsymbol{w}_a^{(iv)}(x) = \boldsymbol{F}_a\begin{bmatrix}f_0(x)\\ f(x)\end{bmatrix} \tag{27}$$

Note that, by including the initial deflection $w_0(x)$ in the coordinate, Eq. (12) can be rewritten as in Eq. (27). Method 2 has broader applications than method 1 in the sense that it can directly accommodate the effect of model reduction with a nonzero initial displacement as can be seen in the subsequent sections.

Wave Propagation

In the preceding section, the matrix differential equation is derived based on the time domain finite element analysis. As a result, a fourth-order ordinary differential matrix equation is obtained in the spatial domain. In this section, spatial propagation due to the initial displacement of the beam will be investigated under various geometric and natural boundary conditions. It can be seen from the differential equation obtained so far that a certain property propagates bilaterally through a medium in a spatial domain, departing from one end of the beam to the other end of the beam. This is called the wave because the derived closed-form matrix differential equation preserves the characteristics of the original second-order wave equation of the beam. Therefore, it can be analyzed that the physical properties such as transverse deflection, transverse angle, moment, and shear are moving with waves through a medium. Moreover, when a certain motion is generated at a certain location of a beam, then the aforementioned four properties travel throughout the beam to both directions following the rule expressed as the equation obtained in the preceding section without violating the geometric and natural boundary conditions.

Consider the wave propagation equation using method 1 again. Equation (12) can be transformed into the equivalent first-order state-space equation by defining $y(x) \equiv [w(x)\ \theta(x)\ M_b(x)\ V_s(x)]^T$ as follows:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = Ay(x) + Bf(x) + B_0 w_0(x) \tag{28}$$

where $\theta(x)$, $M_b(x)$, and $V_s(x)$ denote transverse angle, bending moment, and shear force, respectively, and

$$A = \begin{bmatrix} 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & (1/EI)\boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{I} \\ EI\boldsymbol{K}^{-1}\boldsymbol{M} & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ EI\boldsymbol{K}^{-1}\boldsymbol{F} \end{bmatrix} \tag{29}$$

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ EI\boldsymbol{K}^{-1}\boldsymbol{F}_0 \end{bmatrix}$$

where $\boldsymbol{I}$ denotes the identity matrix whose size is equal to the number of nodes generated by the finite element analysis in the time domain. The solution of Eq. (28) can be expressed as follows:

$$y(x) = e^{A(x-x_0)}y(x_0) + \int_{x_0}^{x} e^{A(x-\chi)}[Bf(\chi) + B_0 w_0(\chi)]\,\mathrm{d}\chi \tag{30}$$

Equation (30) can be transformed into the equation in the wave domain using the modal matrix Φ of the system matrix A. Introducing the coordinate transformation $y(x) = \Phi\eta(x)$, Eq. (30) can be converted into the wave domain equation as follows:

$$\frac{\mathrm{d}\eta(x)}{\mathrm{d}x} = \Lambda\eta(x) + \tilde{B}f(x) + \tilde{B}_0 w_0(x) \tag{31}$$

where the following relations are satisfied:

$$\Phi^{-1}A\Phi = \Lambda, \qquad \tilde{B} = \Phi^{-1}B, \qquad \tilde{B}_0 = \Phi^{-1}B_0 \tag{32}$$

Wave properties are described as a function of x using the solution of Eq. (31):

$$\eta(x) = e^{\Lambda(x-x_0)}\eta(x_0) + \int_{x_0}^{x} e^{\Lambda(x-\chi)}[\tilde{B}f(\chi) + \tilde{B}_0 w_0(\chi)]\,\mathrm{d}\chi \tag{33}$$

Assume that 1) the wave is generated by initial deflection and 2) a force actuator is located at the distance x_f from the root of the beam; then Eq. (33) can be represented as

$$y(L) = \Phi\, e^{\Lambda L}\Phi^{-1}y(0) + \Phi\, e^{\Lambda(L-x_f)}\Phi^{-1}B + \Phi H\tilde{B}_0 \tag{34}$$

where

$$H = \int_0^L e^{\Lambda(L-\chi)}w_0(\chi)\,\mathrm{d}\chi$$

Let us define

$$\Omega \equiv \Phi\, e^{\Lambda L}\Phi^{-1}, \qquad \mu \equiv \Phi\, e^{\Lambda(L-x_f)}\Phi^{-1}B + \Phi H\tilde{B}_0$$

The partition of Ω and μ gives

$$\begin{bmatrix} w(L) \\ \theta(L) \\ M_b(L) \\ V_s(L) \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} w(0) \\ \theta(0) \\ M_b(0) \\ V_s(0) \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \tag{35}$$

Equation (35) can be divided into two groups, and for the fixed-free beam the following boundary conditions must be satisfied at both ends:

$$w(0) = 0, \qquad \theta(0) = 0$$
$$M_b(L) = 0, \qquad V_s(L) = 0 \tag{36}$$

Using Eq. (36) in Eq. (35) yields

$$\begin{bmatrix} w(L) \\ \theta(L) \end{bmatrix} = \Omega_{12}\begin{bmatrix} M_b(0) \\ V_s(0) \end{bmatrix} + \mu_1 \tag{37}$$

$$0 = \Omega_{22}\begin{bmatrix} M_b(0) \\ V_s(0) \end{bmatrix} + \mu_2 \tag{38}$$

Therefore, we can obtain the following useful relations for initial and final conditions of the residuals, and these relations can be used to guess the unspecified initial/final conditions:

$$\begin{bmatrix} M_b(0) \\ V_s(0) \end{bmatrix} = -\Omega_{22}^{-1}\mu_2 \tag{39}$$

$$\begin{bmatrix} w(L) \\ \theta(L) \end{bmatrix} = -\Omega_{12}\Omega_{22}^{-1}\mu_2 + \mu_1 \tag{40}$$

Model Reduction Using Time-Based Modal Coordinates

In the preceding sections, we investigated how to build matrix differential equations on spatial propagation of second-order wave equations. The dimension of the matrices using methods 1 and 2 is the same as the number of temporal nodes. If the system dynamics is very fast, then it requires a huge number of finite elements. This will also result in a drastic size of the matrix, and therefore numerical accuracy and computational time may emerge as a big problem to be worked out. This problem is closely related to the frequency spectra contained in the system dynamics and input applied to the system. Therefore, it is extremely important to reduce the size of the matrix to handle the problem in the time domain finite element analysis. Much work has been done on model reduction in the field of structural analysis and linear control theory. Model reduction in the time domain proposed herein is the extension of space domain equivalence. In this study, we will consider the problem that satisfies the following initial/final conditions:

$$\dot{w}(x, 0) = 0, \qquad \dot{w}(x, t_f) = 0 \tag{41}$$

Now we will show that the spatial propagation equation satisfies the characteristics of the self-adjoint system.[19] Consider the partial differential equation of a beam

$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{42}$$

Separation of variables gives

$$w(x, t) = Y(x)F(t) \tag{43}$$

By substituting Eq. (43) into Eq. (42), we obtain the following two ordinary differential equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2}\right] - \omega^2\rho(x)Y(x) = 0 \tag{44}$$

$$\frac{\mathrm{d}^2F(t)}{\mathrm{d}t^2} + \omega^2F(t) = 0 \tag{45}$$

In consideration of Eq. (45), the following system can be shown to be self-adjoint in the time domain:

$$\boldsymbol{M}w(x) + \boldsymbol{K}w^{(iv)}(x) = \boldsymbol{F}f(x) + \boldsymbol{F}_0 w_0(x)$$

Let $\Pi = \mathrm{d}^2/\mathrm{d}t^2$ and $\Xi = -1$; then

$$\int_0^{tf} F_r(t)\Pi[F_s(t)]\,\mathrm{d}t = \int_0^{tf} F_r(t)\ddot{F}_s(t)\,\mathrm{d}t = -\int_0^{tf}\dot{F}_r(t)\dot{F}_s(t)\,\mathrm{d}t \tag{46}$$

Note that an additional term vanishes because eigenfunctions of the time domain finite element analysis satisfy the following initial and terminal boundary conditions:

$$\dot{F}_i(t)|_{t=0} = 0, \qquad \dot{F}_i(t)|_{t=t_f} = 0 \tag{47}$$

Similarly, we have

$$\int_0^{t_f} F_s(t)\Pi[F_r(t)]\,\mathrm{d}t = -\int_0^{t_f} \dot{F}_s(t)\dot{F}_r(t)\,\mathrm{d}t \tag{48}$$

Equations (46) and (48) yield the following relation:

$$\int_0^{t_f} F_r(t)\Pi[F_s(t)]\,\mathrm{d}t = \int_0^{t_f} F_s(t)\Pi[F_r(t)]\,\mathrm{d}t \tag{49}$$

And naturally

$$\int_0^{t_f} F_r(t)\Xi[F_s(t)]\,\mathrm{d}t = \int_0^{t_f} [-F_s(t)F_r(t)]\,\mathrm{d}t$$

$$= \int_0^{t_f} F_s(t)\Xi[F_r(t)]\,\mathrm{d}t \tag{50}$$

From Eqs. (49) and (50), we can see that the propagation equations (12) and (27) are self-adjoint.

Eigenfunctions $F_r(t)$ and $F_s(t)$ of the system generated by finite element analysis in the time domain satisfy the following equation:

$$\int_0^{t_f} F_r(t)\Xi[F_s(t)]\,\mathrm{d}t = c_r\delta_{rs}, \qquad r, s = 1, 2, \ldots \quad \text{for} \quad \lambda_r \neq \lambda_s \tag{51}$$

if eigenfunctions are normalized such that

$$\int_0^{t_f} F_r(t)\Xi[F_r(t)]\,\mathrm{d}t = \delta_{rs}, \qquad r = 1, 2, \ldots \tag{52}$$

Then we also have

$$\int_0^{t_f} F_r(t)\Pi[F_s(t)]\,\mathrm{d}t = \lambda_r\delta_{rs}, \qquad r, s = 1, 2, \ldots \tag{53}$$

Therefore, orthogonality holds for the time-based natural modes of the propagation equation.

Let us consider the wave propagation equation, Eq. (12), again:

$$-\boldsymbol{M}w(x) + \boldsymbol{K}w^{(iv)}(x) = \boldsymbol{F}f(x) + \boldsymbol{F}_0 w_0(x)$$

The homogeneous equation is

$$-\boldsymbol{M}w(x) + \boldsymbol{K}w^{(iv)}(x) = 0 \tag{54}$$

By assuming a harmonic solution $w(x) = \psi\, e^{i\omega_k x}$, we obtain the eigenvalue problem in time domain finite element analysis as follows:

$$\boldsymbol{M}\psi_k = \omega_k^4 \boldsymbol{K}\psi_k \tag{55}$$

Therefore, an equivalent eigenvalue concept can be applied to the system in the time domain finite element analysis. Note that the wave propagation equation is self-adjoint and that the eigenvectors are orthogonal. Therefore, using the modal matrix, the full-order wave propagation equation can be approximated to the reduced-order model. The order-reduction procedure in the time domain finite element analysis is analogous to that in conventional FEM.

Numerical Examples

In this section, we present numerical examples to verify the proposed method and to validate the effect of model reduction. A modal truncation approach is used to reduce the order of the model using a time-based modal coordinate. The time increment at each time step can be varied at every finite interval throughout the final time to accommodate the abrupt changes of dynamic response and/or actuator input. A linear, quadratic, or cubic function can be selected for a shape function. A higher-order shape function can enhance the accuracy of the time response at the expense of enlarging the model size. A simple pendulum problem is considered to simulate the open-loop and closed-loop responses, and a cantilever beam model is also investigated. Numerical results using the time domain finite element analysis are compared with the conventional FEM results.

Simple Pendulum

In this example, the motion of a simple pendulum is investigated using the time domain finite element analysis. Consider a simple pendulum composed of a lumped mass of 2 kg and a weightless bar of length 50 cm. The bar is connected to the ceiling with a joint, and control moment can be exerted to this system by the actuator placed at a joint. The single generalized coordinate θ is the angular displacement of the bar from the vertical, and we will start our pendulum at $\theta = 10$ deg and $\dot{\theta} = 0.0$. Mass and stiffness matrices are constructed using the method mentioned earlier. In this particular problem there is no spatial variable, and the angular displacement of the pendulum depends only on the time variable. Therefore, the time history of the motion can be easily obtained as follows:

$$\theta(t) = [K_a - M_a]^{-1} F_a \begin{bmatrix} f_0(x) \\ f(x) \end{bmatrix}$$

Numerical simulation is conducted to compare the results using the time domain finite element analysis. Figures 1–3 show the open-loop results of the pendulum. The pendulum oscillates from the initial angle with a period of 1.45 s, and the result of the time domain finite element analysis corresponds exactly with direct numerical integration, as can be seen in Fig. 1. Figure 2 shows the comparative result using the full model and the reduced-order model. The first 9 modes of temporal coordinates of 143 modes are retained in the reduced-order model to describe the motion of the pendulum, and we can see that 9 time-based modal coordinates can successfully reconstruct the time histories of the pendulum.

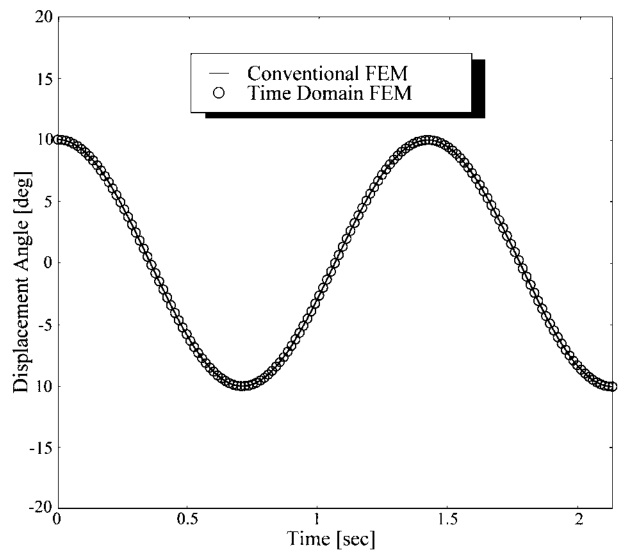


Fig. 1 Initial condition response using the full model.

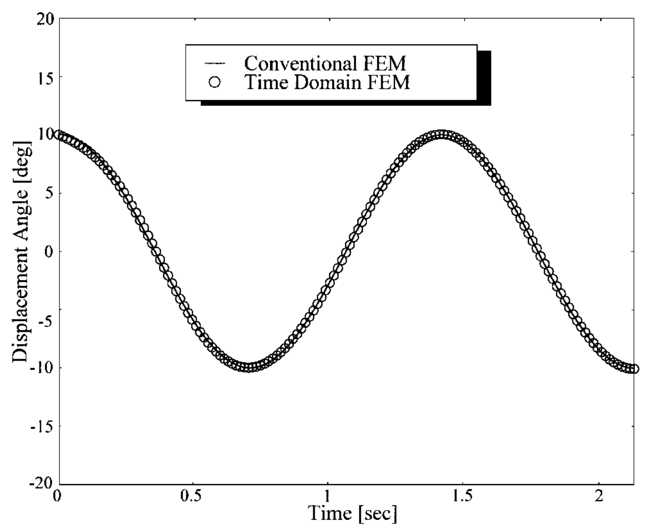


Fig. 2 Initial condition response using the reduced model.

Table 1 Physical properties of the beam

Description	Value
Length, m	0.8100
Width, m	0.0635
Thickness, m	0.0010
Density, kg/m	0.3556
Young's modulus, N/m^2	7.6×10^{10}

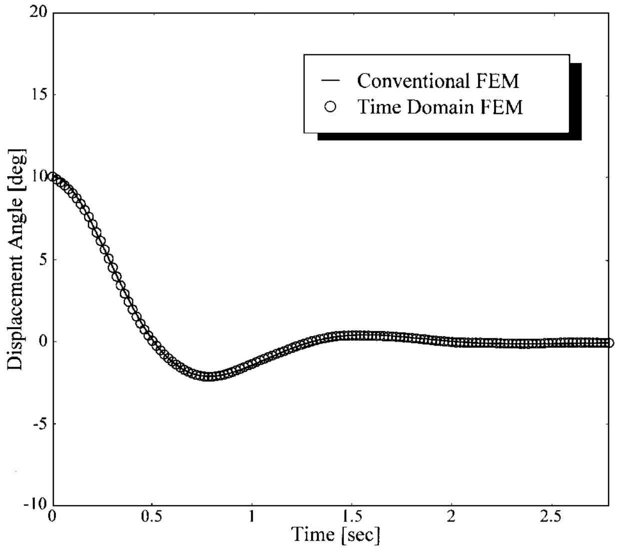


Fig. 3 Forced response using the reduced model.

Because of a restriction on the final condition, the final time should be set at a multiple of the half-period so that the velocity of the pendulum may be zero at the final time. Figure 3 shows the open-loop responses where the forcing history is obtained by a linear quadratic control input to regulate the system. The reduced-order model response shows a good performance in the time domain finite element analysis. Moreover, because the motion of the pendulum is regulated after a certain time by the regulating input history, it is guaranteed that the velocity of the pendulum vanishes after that. This removes all interferences that restrict the application of the proposed methods, and we only have to take the time when the motion is regulated.

Cantilever Beam

In this example, the time response of a cantilever beam under zero initial displacement is investigated. The physical properties of the beam are listed in Table 1. A point force input is applied to the structure at the tip, and a sinusoidal input is imposed to excite the structure:

$$u(t) = \sin \tfrac{3}{2}\pi t \tag{56}$$

with the known external disturbance

$$\mathrm{d}(t) = \sin 1.4\pi t + \cos 3\pi t \tag{57}$$

A total of 49 finite elements in the time domain and a second-order shape function are used. After constructing matrices, we transform the equation of motion into the time-based modal coordinate. The first 15 modes are used to build the time response of the beam. Figure 4 shows the tip deflection histories of the beam. Figure 4 shows the conventional FEM results and the time domain finite element analysis results for comparisons. It shows that the results obtained via the time domain finite element analysis correspond exactly with the FEM results. It can be seen from Fig. 4 that a one-fifth-order reduction of the model can follow the response very well.

In this example, the performance of model reduction in the time domain finite element analysis with a nonzero initial spatial displacement is also investigated using both methods 1 and 2. A static point load of $1N$ is applied to the tip of the beam to enforce the initial deflection of the beam. Initial spatial displacement yields

$$w_0(x) = (Lx^2/2EI) - (x^3/6EI) \tag{58}$$

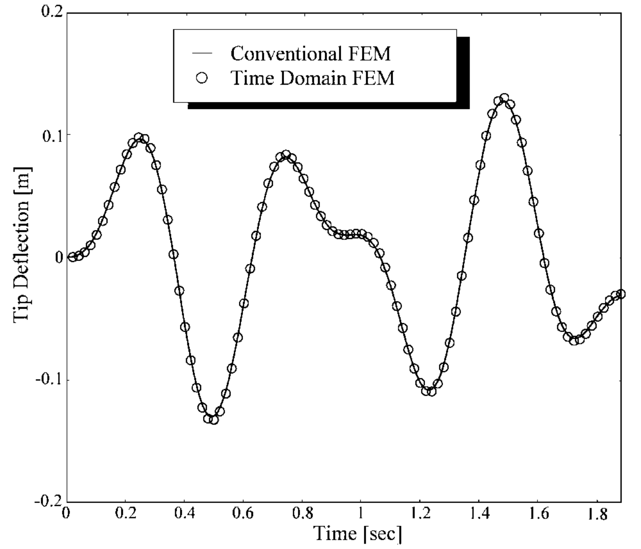


Fig. 4 Forced response with external disturbance using the reduced model.

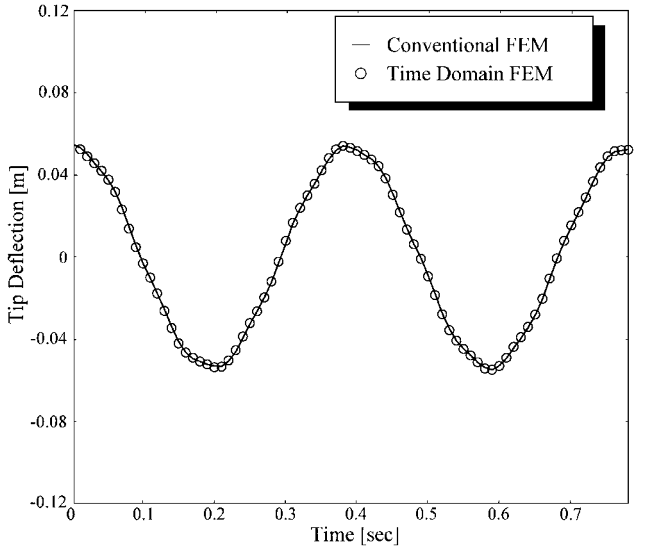


Fig. 5 Initial condition response using method 1 and the full model.

An actuator input is imposed to suppress the vibration of the structure using a forced open-loop input history dictating that of the linear quadratic regulator. It is verified that the motion of the beam is regulated within 2 s. The motion of the beam at the tip can be obtained using Eqs. (33) and (34) for the system obtained by method 1:

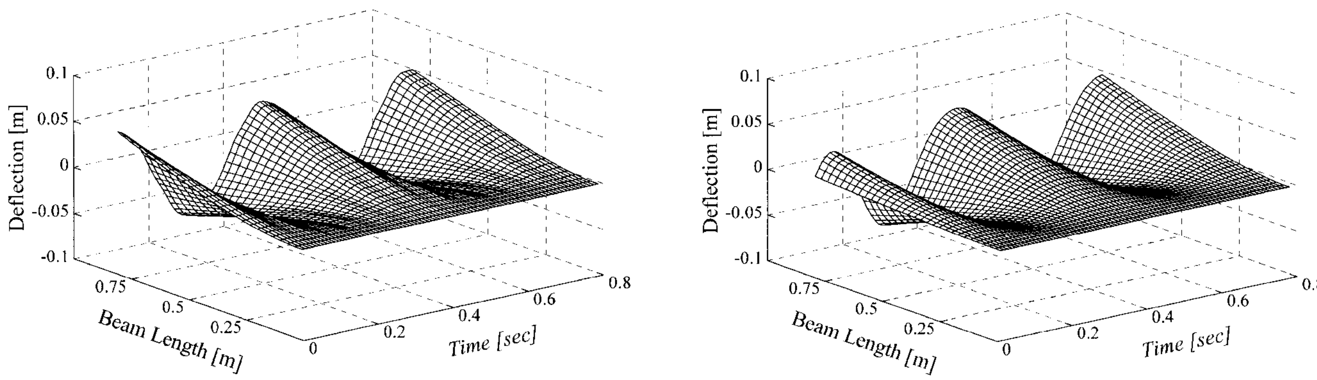
$$\eta(L) = e^{\Lambda L}\eta(0) + e^{\Lambda(L-x_f)}\tilde{B} + \int_0^L e^{\Lambda(L-\chi)}\left(\frac{L\chi^2}{2EI} - \frac{\chi^3}{6EI}\right)\mathrm{d}\chi\,\tilde{B}_0 \tag{59}$$

$$y(L) = \Phi\eta(L) = \Phi\, e^{\Lambda L}\Phi^{-1}y(0) + \Phi\, e^{\Lambda(L-x_f)}\Phi^{-1}B + \Phi H\tilde{B}_0 \tag{60}$$

where

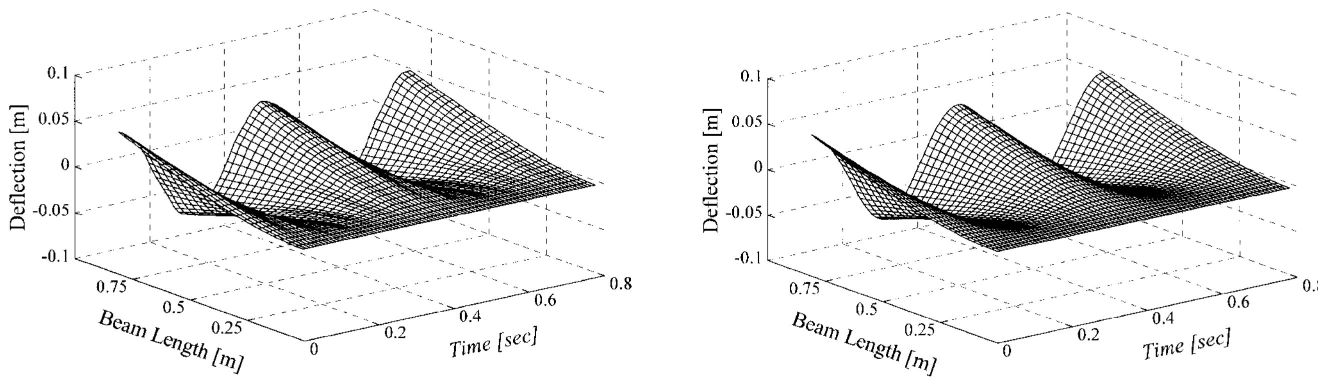
$$H_{ii} = \frac{1}{EI}\left(-\frac{L^3}{3\lambda_i} - \frac{L^2}{2\lambda_i^2} + \frac{1}{\lambda_i^4}\right) + \frac{e^{\lambda_i L}}{EI}\left(\frac{L}{\lambda_i^3} - \frac{1}{\lambda_i^4}\right) \tag{61}$$

Figures 5 and 6 show the results using the procedures with the full model and reduced model, respectively, when the system is modeled using method 1. Figure 5 shows that the formulation using method 1 can be used effectively to simulate dynamic systems with a nonzero initial condition when a full model is used. However, the model reduction results are not very satisfactory, as seen in Fig. 6. This is because the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ obtained from method 1 do not contain the information on initial displacement, and this generates



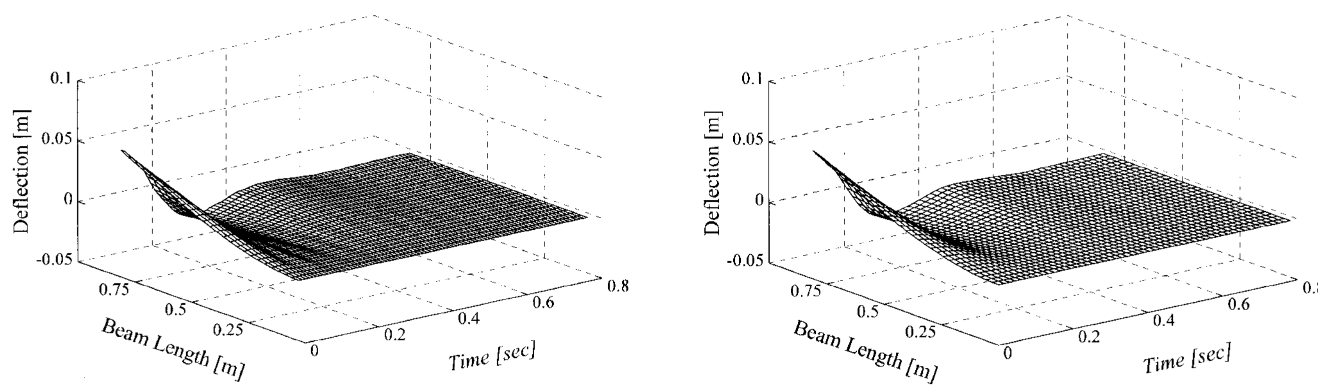
Conventional FEM **Time domain FEM**

Fig. 6 Three-dimensional mesh plot for the initial condition response using method 1 and the reduced model.



Conventional FEM **Time domain FEM**

Fig. 7 Three-dimensional mesh plot for the initial condition response using method 2 and the reduced model.



Conventional FEM **Time domain FEM**

Fig. 8 Three-dimensional mesh plot for the forced response using method 2 and the reduced model.

a discrepancy with FEM in the vicinity of the initial time. Consequently, the first few modes can describe a zero initial status very well, as seen in Fig. 4, but not for the nonzero initial displacement condition. These results indicate that there should be another matrix construction to simulate a nonzero initial displacement from the beginning. Figures 7 and 8 show a three-dimensional mesh plot of the beam response using method 2 compared with FEM. They show complete agreement with the results obtained by FEM, at every time at every point of the beam, in both the initial condition response and forced response with the linear-quadratic-based regulating input. It can be verified from the figures that method 2 can simulate the effect of initial displacement exactly and can accommodate model reduction perfectly for both initial condition/forced responses.

Conclusion

In this study, dynamic analyses of systems are investigated using the time domain finite element analysis when initial and final conditions are prescribed. A new method is developed to describe the motion of the system, and it is extended to accommodate the initial displacement as a state variable. The spatial propagation equation is built up, and two-point boundary conditions are used to derive the initial conditions of the physical properties at one end of the beam. It can be seen from the equation that the physical properties transfer through the beam element without violating boundary conditions at both ends. Modal domain analysis is introduced to reduce the size of the matrix constructed using the time domain finite element analysis. It has been verified that the time-based modal coordinate system stays within the region of a self-adjoint system. This enables the reduction of modes and finally can reduce the size of the system significantly. This also means that a small number of modal coordinates can describe the complex system response well. Several numerical examples are shown to verify the suggested methods and to validate the effect of significant model reduction. The consequences of the suggested methods open another realm of various structural design techniques by releasing a degree of freedom called spatial distribution.

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G. M. Faeth
Editor-in-Chief